In the aftermath of the failure of the no-general propositions theory of *InS*, Russell hurriedly wrote "Mathematical Logic as Based on the Theory of Types" (*ML*). Cocchiarella (1980) seems fist to have noticed that *ML* is the first appearance of a genuinely ramified substitutional theory of propositions, replete with an *ad hoc* Axiom of Reducibility for propositions. This went largely unnoticed by Church and others because the lion's share of the work of *ML* employs what it calls the "convenience" of the language of bindable object-language predicate variables and the scheme for the translation from its bindable predicate variables into the language of substitution was only very briefly sketched. Since Russell's work on substitution was largely hidden and appeared only in French, it was easy to lose sight of the fact that he was advocating a new ramified version of his substitutional theory.

In *ML*, Russell admitted that his ontology of *orders* of propositions is untoward and that the work wants revision even if the broad outline is on the right track. The substitutional theory of *ML* now embraces orders of general propositions. As a result, it *emulates* a ramified typed regimented ^{cp}Logic of universals with a reducibility axiom. The orders of propositions are codified syntactically by *order* subscripts on variables and terms of the language of the substitutional theory. With propositions ramified into orders, violations of Cantor's diagonal power theorem cannot arise from functions involving propositional identities in the ramified substitutional theory. The p_o/a_o paradox is blocked, while at the same time enabling the introduction reducibility axioms for propositions which, unlike the unbridled mitigating axioms of *InS*, consistently preserve Cantor's power-theorem, the theorem of mathematical induction and the development of mathematics in general.

Russell's 1907 manuscript "On Types" summarizes a year-long struggle over how to introduce such order subscripts in a philosophically tenable, albeit *ad hoc* way. Using subscripts, we can nominalize a formula of the language of the ramified substitutional theory of *ML* to make a term $\{A\}_v$ in accordance with the following rule. If *n* is the highest order index on any propositional variable occurring in the *wff* *A*, then *v* = *n +1* if the variable is bound and *v* = *n* if the variable is free. In manuscripts, Russell experimented with all sorts order assignments. When the dust settled, the new regimentation on the formulas of the language of substitution demands the following:

$$p_m \frac{x_n}{a_n} ! \, q_m.$$

In the substitutitional theory, there is no such thing as type since the theory of simple-types of universals is emulated in a "no-comprehension principles for universals" theory which builds simple type distinctions into the number of substitutions involved. But in *ML*, with the substitutional theory now regimented by order indices on its variables, what is emulates is no longer a simple impredicative type theory comprehending universals. For Russell, the "predicative" substitutional matrices, are ones whose order indices step downward one at a time. For example, these are such matrices as

p_1/a_o,

$q_2 \,/\, p_1, a_o$

$j_3/\, q_2, p_1, a_o$

and so on. The chart (below) shows the relationship between order\type indices on the object-language bindable predicate variables in *Principia* and the substitutional emulation of them in the ramified substitutional theory of *ML*.

Simple type	Type + order	Substitutional matrix + order
$\varphi^{(o)}$	$^1\varphi^{(o/o)}$	p_1/a_o
	$^2\varphi^{(o/o)}$	p_2/a_o
$\varphi^{((o))}$	$^2\varphi^{(1/(o/o))}$	$q_2/\, p_1, a_o$
	$^3\varphi^{(1/(o/o))}$	$q_3/\, p_1, a_o$
$\varphi^{(((o)))}$	$^3\varphi^{(2/(1/(o/o)))}$	$s_3 \,/\, q_2, p_1, a_o$
	$^4\varphi^{(2/(1/(o/o)))}$	$s_4 \,/\, q_2, p_1, a_o$
	$^4\varphi^{(3/(1/(o/o)))}$	$s_4 \,/\, q_3, p_1, a_o$
	$^4\varphi^{(3/(2/(o/o)))}$	$s_4 \,/\, q_3, p_2, a_o$

With this in place, the substitutional technique *emulates* a ramified type theory of attributes. For example, we get the following as theorems. We have

$$(\exists p_n, a_o)\,(x_o)(\, p_n \frac{x_o}{a_o} \equiv \{Ax_o\}_n),$$

where p_n and a_o are not free in the *wff A*. For the next type, we have the following theorem which is proscribed by Russell's decision concerning $p_m \frac{b_u}{a_v}\, q_m$. We have:

$$(\exists s_m, t_1, w_o)(\, s_m \frac{p_1,a_o}{t_1,w_o} \equiv_{p_1,a_o} \equiv \{\mathrm{A}[p_1, a_o]\}_m\,)$$

where s_m, t_1, and w_o are not free in the *wff* A. The substitutional theory of *ML* next adds axioms (not axiom schemas) of reducibility such as the following

$$(p_m, b_o)(\exists\, p_1, a_o)(\, p_1 \frac{x_o}{a_o} \equiv_{x_o}\ p_m \frac{x_o}{b_o}).$$

$$(s_m, t_1, w_o)(\exists s_2, t_1, w_o)(\, s_2 \frac{p_1,a_o}{t_1,w_o} \equiv_{p_1,a_o}\ s_m \frac{p_1,a_o}{t_1,b_o}).$$

and so on. We now see how the orders of propositions together with the technique of substitution emulates a *ramified* type theory of attributes in intension. It is this theory that mandates an axiom of reducibility to offset the deleterious effect of orders in extensional contexts. This is quite different from the system of *InS* where, contrary to the interpretation of Hylton (1980), there are no orders, no general propositions and no ramification at all. These axioms of reducibility of *ML* succeed in preserving Cantor's work. At the same time, they cannot revive the p_o/a_o paradox. They succeed, where Russell thought the mitigating axioms of *InS* had failed. The axioms of reducibility of *ML* enable the system to recover from the crippling effect that ramification has on the emulation of attributes (and so also classes) in the theory. Given we can accept the axiom, Cantor's work and the logicist development of mathematics is preserved.

We saw that the p_o/a_o paradox is blocked in substitution if we introduce order restrictions into substitution with the requirement that in

$$p_m \frac{x_d}{a_n} ! \, q_m$$

$d = n$. But it is possible to block the paradox with the more modest requirement that $n \geq d$.
Russell was aware that different restrictions concerning substitution of orders might well also work to avoid the p_o/a_o paradox and that his particular restriction on substitution seems not to be motivated by any logical analysis of the nature of general propositions. Ultimately, he found the entire situation unsatisfactory and abandoned propositions and orders altogether. This final decision likely came in 1908. In any case, by 1910 and in *Principia,* he abandoned the substitutional theory. But we can easily imagine adopting cumulation so that

$$(p_n)\mathrm{A}p_n \supset \mathrm{A}p_d$$

where $n \geq d$. If we make these changes, the language of Church's *Principia*^{C} can be translated into the language the ramified substitutional theory of "Mathematical Logic" rather faithfully.
Indeed, Church's theory of ramified types of universals actually fits the theory emulated by the ramified substitutional theory of *ML*. And Church (1976) admits that he was working predominantly from the ideas of *ML*, rejecting many of the comments of Whitehead and Russell's introduction to *Principia* as untoward (if coherent). We saw that it is likely that Russell intended in *ML* that the "predicative" substitutional matrices are ones whose order indices step downward one at a time. But in the above variant that parallels the Church cumulative of orders in *Principia*^{C}, the predicative matrices would naturally be those that are substitutional translations of Church's notion that the *predicative* predicate variables are just those whose level is 1. Hence, it allows the following (which gives Church's ramified-type regimented predicate variable on the left and its analog Russellian substitutional matrix on the right):

$$\varphi^{((o)/2)/1} \qquad q_3 \,/\, p_2, a_o.$$

Church regards this as predicative. Note as well that Church allows the following:

$$\varphi^{(o)/2} \qquad p_2/a_o$$

$$\varphi^{((o)/1)/2} \qquad q_3/p_1, a_o\ .$$

Nothing in the substitutional theory of *ML* can prevent such matrices. Church's system of ramified types best fits Russell's remarks in *ML* once appropriate modifications are made for cumulative substitutions. In summary, Church's *Principia*^{C} is closest to what *Principia* would have been had Russell not abandoned his ramified substitutional theory.