In what follows, the article sets out two main interpretations of the formal logic of *Principia Mathematica*, one aims to be historical which will be called $Principia^L$ and the other is Church's interpretation which shall be called $Principia^C$. Church's interpretation is vastly more complex. It is best explained in terms of adjustments and complications added on to $Principia^L$. Moreover, we shall see that Church arrived at his interpretation by looking, not at *Principia Mathematica*, but at a work Russell completed in 1907 called "Mathematical Logic as Based on the Theory of Types"—a work, unbeknownst to Church, in which Russell still embraced his substitutional theory of propositional structure. Thus it is much more natural, in spite Church's popularity with formal logicians, to begin with the simple type theory of $Principia^L$.

The historically oriented $Principia^L$ maintains that letters such as *x, y, z* are *Principia Mathematica's* object language individual variables. Letters *f!* φ!, ψ!, θ!, and χ! are the predicate variables of the formal language. In contrast, letters such as *f*, φ, ψ, θ, and χ are used schematically for *wffs*, that is, well-formed formulasThey are not predicate variables of the formal object-language. In short, the exclamation sign '!' marks the predicate variables of the formal system. If we were to restore suppressed type-indices to the predicate variables, we find that all and only variables in *Principia Mathematica* are predicative, that is, the order is the order of the simple-type symbol. If type indices were restored the exclamation would not be needed and neither would differences between predicate and individual variables be needed. That is to say, we would find, for example, x^t and $x^{(t)}$ and $x^{(t,t)}$ and $x^{(t,(t))}$ and so on.

The intended formal grammar of $Principia^L$ is just that of simple type theory. There are many passages of *Principia Mathematica* that support this interpretation, but we shall not delay to present detailed evidence here (for detailed historical evidence for this interpretation of *Principia Mathematica* see Landini 1998) Church knew that his interpretation of *Principia Mathematica* was not historically accurate—that he was giving an improved formal system which he thought was consistent with the spirit of Russell and Whitehead's work (see, Church, 1976, p. 291). For the present, all we need is to note that Whitehead and Russell explicitly rule out binding variables that do not have the exclamation. *Principia Mathematica* is quite explicit in adopting no object-language bindable variables that do not have the exclamation (that is, are not predicative). As we shall see, a predicative index on a bindable predicate variable is one whose order in not the order of their simple type. In *Principia Mathematica*, all and only genuine variables are predicative. Thus, bindable predicate variables φ!, ψ!, *f* !, *g*! χ! and the like always come with the exclamation (shriek !). (*Principia Mathematica* permits "(φ)(….φ!...)" and "(∃φ)(…φ!...)" instead of "(φ!)(….φ!...)" and "(∃φ!)(…φ!...)" which it admits is more formally correct (*Principia*

Mathematica, p. 165). When letters φ, ψ, *f* , *g*, χ occur *without* the shriek they are schematic for *wffs.* They cannot be bound because they are not object-language predicate variables.

The historical *Principia Mathematica* allows no non-predicative variables in its grammar. Thus, ramification is not coded into its syntax. According to $Principia^L$ ramification is a product of the intended nominalistic (modern substitutional) semantics Whitehead and Russell offered for the genuine (predicative) predicate variables of the formal system. That is, by the lights of its intended nominalistic semantics, its axiom schema for comprehension is unwarranted. It is not valid in the nominalistic semantics.

With this in mind, we can present the formal logic of *Principia Mathematica* as follows. The primitive signs of the language of *Principia Mathematica* are *v*, ~, (,), ' (prime), and ∃. The symbol ∀ for universal quantification is not used in *Principia Mathematica*, but we shall use it in what follows since it is very convenient in notations of type theory. Predicate variables and individual variables come with order or type symbols. The individual variables are x_1^o, x_2^o ,…, x_n^o (informally x^o, y^o, z^o), and the predicate variables are x_1^t, x_2^t ,…, x_n^t (informally φ^t, ψ^t, θ^t). A type symbol of simple type theory is defined recursively as follows:

(i) *o* is a type symbol.
(ii) If $t_1,\ldots, t_n$ are type symbols, then $(t_1,\ldots, t_n)$ is a type symbol.
(iii) There are no other type symbols.

It is useful to also have the notion of the *order* of a simple type symbol. This can be recursively defined as follows:

(i) The type symbol *o* has order 0.
(ii) A type symbol $(t_1,\ldots, t_n)$ has order *m*+1 if the highest order of the type symbols $t_1,\ldots, t_n$ is *m*.

There is an easy parenthesis counting technique to determine the order of a simple type symbol (see, Hatcher 1980, p. 107). For each left-hand parenthesis *add* 1 and for each right-hand parenthesis *subtract* 1. The highest number reached is the order of the simple type symbol. Thus, for instance, the order of ((o),o) is 2. Thus, this weak notion of order is already coded into the simple type symbol and plays no role in the formal system whatsoever. The natural interpretation of types often renders x^o as a variable for an individual (lowest type) and construes these as concrete particulars. But, this is not the interpretation Whitehead and Russell envisioned since Russell was on record holding that minds can be *acquainted* with universals, and the *acquaintance* relation and universals are type-free individuals. In truth, the two may not have wholly agreed on what is the best interpretation of their formal system. In any case, on a Realist interpretation of the object-language bindable predicate variables, one would say that the predicate variable $\varphi^{(o)}$ is for a property of individuals; the predicate variable $\varphi^{((o))}$ is for a property of a property of an individual, and so on. The predicate variable φ$^{(o.o)}$ is for a dyadic relation of individuals. The predicate variable φ$^{((o).o))}$ is for a dyadic relation between individuals. Non-homogeneous relations are allowed. The variable φ$^{((o).o))}$ is for a dyadic relation between an individual and an property of an individual. The atomic *wffs* are of the form,

$$\varphi^{(t_1,\ldots t_n)}(x_1^{t_1},\ldots x_n^{t_n})$$

The practice of typical ambiguity is to suppress type indices on the variables under conventions of restoration. Bindable object-language predicate variables shall be φ!, ψ!, *f!* , *g!* χ! , etc., always with the exclamation '!'. When letters occur *without* the shriek they are schematic for *wffs*. Thus, for example φ*x* indicates a *wff* in which the variable '*x*' occurs free.

The interpretation of the formal logic of *Principia Mathematica* is made difficult by the fact that the grammar of the system is not set out formally and it is not given with type indices on the individual and predicate variables. Instead, type indices are suppressed under conventions of restoration. Once type indices are dropped, the genuine bindable object-language require conventions governing their employment and restoration. Some of the numbered entries of *Principia* are attempts to explain conventions. It is important that they not be conflated with theorems of the formal system itself.

Some of the conventions are obvious. For example, repetitions of an *individual* variable or *predicate* variable in a given line of a proof are understood as uniformly assigned the same simple type indices. Moreover, repetitions throughout a given proof are to be assigned the same simple type indices. This is corroborated, for example, when *Principia Mathematica* writes the following:

> *3.03 Given any two asserted elementary propositional function "├. φ*p*" and "├. ψ*p*" whose arguments are elementary propositions , we have ├. φ*p* **.** ψ*p*.

The uniformity of restoration is not a consequence of *3.03, but of a convention of how to read variable tokens of the same kind. For instance, let the variables be *x*, θ! and φ!, and assign to the schematic *p* of *3.03 the *wff* φ!*x, that is,* $\varphi^{(t)}x^t$. Thus we have:

n. ~φ!*x*   *n*. $\sim\varphi^{(t)}x^t$

n+1. θ!*x* ⊃ φ!*y*   *n*+1. $\theta^{(t)}x^t \supset \varphi^{(t)}y^t$

n+2. ~φ!*x* • (θ!*x* ⊃ φ!*y*)   *n*+2. $\sim \varphi^{(t)}x^t \bullet (\theta^{(t)}x^t \supset \varphi^{(t)}y^t)$

The left is typically ambiguous. The right is the proper type restoration. The convention is that type indices be uniformly assigned throughout the proof. The presence of entries such as *3.03 illustrates some of the many difficulties in understanding *Principia Mathematica*. Simple type indices should have been given with a full statement of the grammar and only then are they should they be dropped and restored under explicit conventions.

The *wffs* are the smallest set K containing all atomic *wffs* such that if φ, and ψ are *wffs* in K and x^t is an individual variable free in formula ψ that is quantifier-free, then

~(φ), (φ ∨ ψ), and $(x^t)(\psi x^t)$ are *wffs*. Where *p, q*, and *r* are schematic for quantifier-free formulas, and φ, and ψ are schematic for any *wffs*, quantifier-free or otherwise, the axiom schema are as follows:

*1.2 $p \vee p .\supset. p$
*1.3 $q .\supset. p \vee q$
*1.4 $p \vee q .\supset. q \vee p$
*1.5 $p \vee (q \vee r) .\supset. q \vee (p \vee r)$
*1.6 $q \supset r .\supset. p \vee q .\supset. q \vee p$

*9.1 $\varphi y^t \supset (\exists x^t)\varphi x^t$

where x^t is free for y^t in the *wff* φ.

*9.12 $\varphi y^t \vee \varphi z^t .\supset. (\exists x^t)\varphi x^t$

where x^t, y^t and z^t are all free for one another in the *wff* φ.

*12.1*n* $(\exists f^{(t_1,\ldots,t_n)})(x_1^{t_1}),\ldots,(x_n^{t_n})(\varphi^{(t_1,\ldots t_n)}(x_1^{t_1},\ldots x_n^{t_n}) \equiv \varphi(x_1^{t_1},\ldots x_n^{t_n}))$,

where $f^{(t_1,\ldots,t_n)}$ is not free in the *wff* φ . With its simple type-symbols suppressed, *Principia Mathematica* expresses two cases of this impredicative comprehension axiom schema with the following:

*12.1 $(\exists f)(x)(f!x \equiv_x \varphi x)$

*12.11 $(\exists f)(f!(x, y) \equiv_{x,y} \varphi(x, y)$

Whitehead and Russell then note that instances of any *adicity* are allowed.

Where *p* and *q* are any *wffs*, quantifier-free or otherwise, the inference rules are the following:

*1.1 *Modus Ponens* (MP)
From *p* and $p \supset q$, infer *q*

*9.13 *Universal Generalization* (UG)
From φx^t, infer $(x^t)\varphi x^t$

Switch
From $(x^{t_1})(\exists y^{t_2})\varphi(x^{t_1}, y^{t_2})$ infer $(\exists y^{t_2})(x^{t_1})\varphi(x^{t_1}, y^{t_2})$
where there is a logical particle in the *wff* φ on one side of which all free occurrences of x^{t_1} occur and on the other side of which all free occurrences of y^{t_2} occur.

Definitions include the following:

$x^t = y^t$ =df $(\varphi^{(t)})(\varphi^{(t)}(x^t) \supset \varphi^{(t)}(y^t))$.

Where φ and ψ are any *wffs*, quantifier-free or otherwise, the following are definitions:

φ ⊃ ψ=*df* ~φ ∨ ψ
φ • ψ=*df* ~(~φ ∨ ~ψ)
φ ≡ ψ =*df* (φ ⊃ ψ) • (ψ ⊃ φ)

Principia Mathematica used a dot. Unlike *Principia*'s use of dots for punctuation, we punctuate using dots symmetrically and this makes reading compound *wffs* easy. The greatest number of dots orders the connectives. Where *p* is quantifier-free and not containing x^t,*Principia Mathematica's* definitions include the following:

*9.01 ~(x^t)φx^t =*df* (∃x^t)~φx^t

*9.02 ~ (∃x^t) φx^t=*df* (x^t)~φx^t

*9.03 (x^t)φx^t ∨ *p* =*df* (x^t)(φx^t ∨ *p*)
*9.04 *p* ∨ (x^t)φx^t =*df* (x^t)(*p* ∨ φx^t)
*9.05 (∃x^t) φx^t∨ *p* =*df* (∃x^t)(φx^t∨ *p*)
*9.06 *p* ∨ (∃x^t) φx^t=*df* (∃x^t)(*p* ∨ φx^t)

Where the *wff* φx^{t_1} does not contain y^{t_2} free and the *wff* ψy^{t_2} does contain x^{t_1} free, *Principia Mathematica* has the following definitions:

*9.07 (x^{t_1})φx^{t_1} ∨ (∃y^{t_2})ψy^{t_2} =*df* (x^{t_1})(∃y^{t_2}) (φx^{t_1} ∨ ψy^{t_2})
*9.08 (∃x^{t_1})φx^{t_1} ∨ (y^{t_2})ψy^{t_2} =*df* (∃x^{t_1})(y^{t_2}) (φx^{t_1} ∨ ψy^{t_2})
*9.0*x* (x^{t_1})φx^{t_1} ∨ (y^{t_2})ψy^{t_2} =*df* (x^{t_1})(y^{t_2}) (φx^{t_1} ∨ ψy^{t_2})
*9.0*y* (∃x^{t_1})φx^{t_1} ∨ (∃y^{t_2})ψy^{t_2} =*df* (∃x^{t_1})(∃y^{t_2}) (φx^{t_1} ∨ ψy^{t_2})

We have added the rule *Switch* from section *8 which was Russell's replacement for section *8 added to *Principia Mathematica's* second edition (which is discussed below) and we added *9.*xx* and *9.*yy*, the omission of which seems to be an oversight. This completes the formal system supporting a huge edifice of mathematics.